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Abstract

A new parafermionic algebra, which we call the twisted parafermion algebra, and its corresponding twisted Z-algebra have been recently proposed. In this paper, we give a free boson representation of the twisted parafermion algebra in terms of seven free fields. A new algebraic structure is discovered, which contains a W-algebra type primary field with spin two.

1 Introduction

The notion of parafermions [1] was introduced in the context of statistical models and conformal field theory [2]. Parafermions generalize the Majorana fermions and have found important applications in many areas of physics. From statistical mechanics point of view, parafermions are related to the exclusion statistics introduced by Haldane [3]. In particular, the Z_k parafermion models offer various extensions of the Ising model which corresponds to the $k = 2$ case [4, 5, 6, 7, 8, 9, 10].

The category for parafermions (nonlocal operators) is the generalized vertex operator algebra [11, 12]. The Z_k parafermion algebra was referred to as Z-algebra in [11, 12], and it was proved that the Z-algebra is identical with the $A_1^{(1)}$ parafermion.

The Z_k parafermions proposed in [1] are basically related to the simplest $A_1^{(1)}$ algebra. Various extensions have been considered by many researchers. Gepner proposed a parafermion algebra associated with any given untwisted affine Lie algebra $\mathcal{G}^{(1)}$ [13, 14], which has been subsequently used in the study of D-branes. The operator product expansions (OPEs) and the corresponding Z-algebra of the untwisted parafermions were studied in [15], and a W_3-algebra was constructed from the $SU(3)$ parafermions. In [16] a W_5-algebra was constructed by using the $SU(2)$ parafermions.

More recently, Camino et al extended the Z_k parafermion algebra and investigated graded parafermions associated with the $osp(1|2)^{(1)}$ super algebra [17]. The central charge of the graded parafermions algebra is $c = -\frac{3}{2k+3}$, which implies that for k a positive integer, c is always negative. Thus the graded parafermion theory is not unitary.

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In [18], we found a new type of nonlocal currents (quasi-particles), which were referred to as twisted parafermions. The twisted system contains a bosonic spin-1 field and six nonlocal fields with fractional spins $1-\frac{1}{4k}$ and $1-\frac{1}{k}$, and leads to a new conformal field theory which is different from the known ones. Let us remark that, while fields with conformal dimensions of $1-\frac{1}{4k}$ and $1-\frac{1}{k}$ also appeared in the graded parafermion theory [17], our twisted parafermion algebra is quite different. In particular, our theory is unitary and has a different central charge.

In this paper, we investigate the theory further. We give a free field representation of the twisted parafermion theory in terms of seven free bosons. We discovered a new algebraic structure which contains a spin-2 primary field. This is similar to a W-algebra structure but now the spin is two.

The layout of the paper is as follows. In section 2, we briefly review the twisted parafermion algebra obtained in [18], which will be extensively used in this paper. In section 3, we give some results concerning the twisted parafermion Hilbert space. In section 4, we give a free field representation of the twisted parafermion currents. We present a new W-algebra structure in section 5.

2 Twisted parafermions: a brief review

The twisted parafermion current algebra proposed in [18] reads:

$$\begin{aligned}
\psi_l(z)\psi_{l'}(w)(z-w)^{ll'/2k} &= \frac{\delta_{l+l',0}}{(z-w)^2}+\frac{\varepsilon_{l,l'}}{z-w}\psi_{l+l'}(w)+\cdots,\\
\psi_{\tilde{l}}(z)\psi_{\tilde{l}'}(w)(z-w)^{ll'/2k} &= \frac{\delta_{\tilde{l}+\tilde{l}',0}}{(z-w)^2}+\frac{\varepsilon_{\tilde{l},\tilde{l}'}}{z-w}\psi_{l+l'}(w)+\cdots,\\
\psi_l(z)\psi_{\tilde{l}'}(w)(z-w)^{ll'/2k} &= \frac{\varepsilon_{l,\tilde{l}'}}{z-w}\psi_{\tilde{l}+\tilde{l}'}(w)+\cdots,
\end{aligned} \tag{2.1}$$

where $l,\ l'=\pm 1$ and $\tilde{l},\ \tilde{l}'=\tilde{0},\pm\tilde{1},\pm\tilde{2}$; $\varepsilon_{l,l'}$, $\varepsilon_{\tilde{l},\tilde{l}'}$ and $\varepsilon_{l,\tilde{l}'}$ are structure constants. If we denote ψ_l or $\psi_{\tilde{l}}$ by Ψ_a, then we can rewrite the above relations as:

$$\Psi_a(z)\Psi_b(w)(z-w)^{ab/2k}\equiv\sum_{n=-2}^{\infty}(z-w)^n[\Psi_a\Psi_b]_{-n}, \tag{2.2}$$

where $a,b=\tilde{0},\pm 1,\pm\tilde{1},\pm\tilde{2}$. So we have $[\Psi_a\Psi_b]_l=0$ $(l\geq 3)$, $[\Psi_a\Psi_b]_2=\delta_{a+b,0}$ and $[\Psi_a\Psi_b]_1=\varepsilon_{a,b}\Psi_{a+b}$. Due to the mutually semilocal property between two parafermions, the radial ordering products are multivalued functions. So we define the radial ordering product of (generating) twisted parafermions(TPFs) as

$$\Psi_a(z)\Psi_b(w)(z-w)^{ab/2k}=\Psi_b(w)\Psi_a(z)(w-z)^{ab/2k}, \tag{2.3}$$

which, like the untwisted case [15], is an extension of the fermion (i.e. $ab=1,\ k=1$) and boson (i.e. $k\to\infty$) theories.

For consistency, $\varepsilon_{a,b}$ must have the properties: $\varepsilon_{a,b} = -\varepsilon_{b,a} = -\varepsilon_{-a,-b} = \varepsilon_{-a,a+b}$ and $\varepsilon_{a,-a} = 0$. The following solution for the structure constants has been found in [18]:

$$\varepsilon_{\tilde{1},-\tilde{2}} = \varepsilon_{1,\tilde{1}} = \varepsilon_{-1,\tilde{2}} = \frac{1}{\sqrt{k}}, \quad \varepsilon_{1,-\tilde{1}} = \varepsilon_{\tilde{0},\tilde{1}} = \sqrt{\frac{3}{2k}}, \tag{2.4}$$

and other constants can be obtained by using the symmetry of properties.

For every field in the parafermion theory there are a pair of charges $(\lambda, \bar{\lambda})$, which take values in the weight lattice. We denote such a field by $\phi_{\lambda,\bar{\lambda}}(z, \bar{z})$ [1, 13, 15]. The OPE of Ψ_a with $\phi_{\lambda,\bar{\lambda}}(z, \bar{z})$ is given by

$$\Psi_a(z)\phi_{\lambda,\bar{\lambda}}(w, \bar{w}) = \sum_{m=-\infty}^{\infty} (z-w)^{-m-1-a\lambda/2k} A_m^{a,\lambda}\phi_{\lambda,\bar{\lambda}}(w, \bar{w}), \tag{2.5}$$

where $m \in \mathbf{Z}$ (Ramond sector) for $a = l$ and $m \in \mathbf{Z} + \frac{\mathbf{1}}{\mathbf{2}}$ (Neveu-Schwarz sector) for $a = \tilde{l}$. A special case of (2.2) is

$$\psi_{\tilde{0}}(z)\Psi_b(w) = \frac{\varepsilon_{\tilde{0},b}}{z-w}\Psi_{\tilde{b}}(w) + \cdots,$$

where field $\psi_{\tilde{0}}(z)$ is a spin-1 primary field, which transfers the fields in the Ramond sector to the Neveu-Schwarz sector or vice versa.

The action of $A_m^{a,\lambda}$ on $\phi_{\lambda,\bar{\lambda}}(z)$ is defined by the integration

$$A_m^{a,\lambda}\phi_{\lambda,\bar{\lambda}}(w, \bar{w}) = \oint_{c_w} dz\, (z-w)^{m+a\lambda/2k}\Psi_a(z)\phi_{\lambda,\bar{\lambda}}(w, \bar{w}), \tag{2.6}$$

where c_w is a contour around w. Let A_a and B_b be two arbitrary functions of the twisted parafermions with charges a and b, respectively. We write OPEs as

$$A_a(z)B_b(w)(z-w)^{ab/2k} = \sum_{n=-[h_A+h_B]}^{\infty} [A_aB_b]_{-n}(w)(z-w)^n, \tag{2.7}$$

where $[h_A]$ stands for the integral part of the dimension of A. Then the so-called twisted Z-algebra relations and the twisted Jacobi-like identities are,

$$\begin{aligned}
&\sum_{l=0}^{\infty} C^{(l)}_{-p-1+ab/2k}\left[A^{a,\lambda+b}_{m-l-p+q}A^{b,\lambda}_{n+l+p-q} + (-1)^p A^{b,\lambda+a}_{n-l-q-1}A^{a,\lambda}_{m+l+q+1}\right] \\
&= C^{(p+2)}_{m+q+1+a\lambda/2k}\delta_{a,-b}\delta_{m,-n} + C^{(p+1)}_{m+q+1+ab/2k}\varepsilon_{a,b}A^{a+b,\lambda}_{m+n} \\
&\quad + \sum_{r=0}^{\infty} C^{p-r}_{m+q+1+ab/2k}[\Psi_a\Psi_b]^{\lambda}_{-r,m+n},
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&\sum_{i=p}^{[h_B+h_C]} C^{(i-p)}_{r-1+ab/2k}[A[BC]_i]_{Q-i}(w) + (-)^r \sum_{j=q}^{[h_C+h_A]} C^{(j-q)}_{r-1+ab/2k}[B[AC]_j]_{Q-j}(w) \\
&= \sum_{l=r}^{[h_B+h_A]} (-)^{(l-r)}C^{(l-r)}_{q-1+ac/2k}[[AB]_lC]_{Q-l}(w),
\end{aligned} \tag{2.9}$$

respectively. These results will be extensively used the sequel.

For the twisted parafermion theory to be a conformal field theory, it is nature to require that the spin-2 terms in the OPEs contain a energy-momentum tensor. It is obvious that in the OPE the spin-2 terms should be contained in $[\Psi_a\Psi_b]_0$. Since the parafermionic charge for the energy-momentum tensor should be zero, so the relevant terms are $[\Psi_a\Psi_{-a}]_0$. Note that $[\Psi_a\Psi_{-a}]_0(z) = [\Psi_{-a}\Psi_a]_0(z)$, we calculate the OPE of $[\Psi_a\Psi_{-a}]_0$ with Ψ_a and $[\Psi_b\Psi_{-b}]_0$. Setting $Q = p = 2,\ q = 1$ and $r = 0$ in (2.9), we have

$$\begin{aligned} [[\Psi_a\Psi_{-a}]_0\Psi_b]_2 &= \delta_{a,b}\Psi_a + \varepsilon_{a,b}\varepsilon_{-a,a+b}\Psi_b \\ &+(1+a^2/2k)\delta_{a,-b}\Psi_{-a} + \frac{ab}{4k}(1-\frac{ab}{2k})\Psi_b. \end{aligned} \tag{2.10}$$

It should be understood that, $\Psi_0(z) = \Psi_{\pm 2}(z) \equiv 0$, that is their parafermionic charges are zero. From the general theory of conformal field theory [2], the conformal dimension of Ψ_a should be $\Delta_a = 1 - \frac{a^2}{4k}$. So we impose the constraints:

$$\sum_a \varepsilon_{a,b}\varepsilon_{-a,a+b} = \frac{6-b^2}{k}, \qquad \sum_a ab = 0, \qquad \sum_a (ab)^2 = 12b^2. \tag{2.11}$$

The structure constants (2.4) satisfy all of these constraints. Then we have

$$\sum_a [[\Psi_a\Psi_{-a}]_0\Psi_b]_2 = \frac{2k+6}{k}\left(1-\frac{b^2}{4k}\right)\Psi_b. \tag{2.12}$$

Choose a proper normalization and write,

$$T_\psi = \frac{k}{2k+6}\sum_a[\Psi_a\Psi_{-a}]_0, \quad [T_\psi\Psi_b]_2 = \left(1-\frac{b^2}{4k}\right)\Psi_b.$$

Repeating the above process, we obtain

$$\begin{aligned} [[\Psi_a\Psi_{-a}]_0\Psi_b]_1 &= \frac{1}{2}\varepsilon_{-a,b}\varepsilon_{a,-a+b}\partial\Psi_b + \frac{1}{2}\varepsilon_{a,b}\varepsilon_{-a,a+b}\partial\Psi_b \\ &+(1+a^2/2k)\delta_{a,b}\partial\Psi_a + (1+a^2/2k)\delta_{-a,b}\partial\Psi_{-a}, \end{aligned} \tag{2.13}$$

or equivalently $[T_\psi\Psi_b]_1 = \partial\Psi_b$. These results can be written in the form of OPEs

$$T_\psi(z)\Psi_b(w) = \frac{1-b^2/4k}{(z-w)^2}\Psi_b(w) + \frac{1}{z-w}\partial\Psi_b(w) + \ldots. \tag{2.14}$$

It follows that the conformal dimension of the twisted parafermion is 1 $(a=\tilde{0})$, $1-\frac{1}{4k}$ $(a=\pm1,\pm\tilde{1})$ or $1-\frac{1}{k}$ $(a=\pm\tilde{2})$, for a given level k. Carrying out a similar program for T, we obtain the OPE:

$$T_\psi(z)T_\psi(w) = \frac{c_{\rm tpf}/2}{(z-w)^4} + \frac{2T_\psi(w)}{(z-w)^2} + \frac{\partial T_\psi(w)}{z-w} + \ldots, \tag{2.15}$$

where $c_{\rm tpf} = 7 - \frac{24}{k+3} = \frac{8k}{k+3} - 1$ is the central charge of the twisted parafermion theory.

3 Twisted Parafermionic Hilbert Space

we now analyze the structure of the Hilbert space of the parafermion theory introduced in the last section. It is obvious that the Hilbert space $\mathcal{H}$ is decomposed by the parafermionic charges. Let us denote the left and right charges as λ, $\bar{\lambda}$, and let $\mathcal{H}_{\lambda,\bar{\lambda}}$ be the subspace of $\mathcal{H}$ with the indicated charge. Then $\mathcal{H}$ is the direct sum of the form:

$$\mathcal{H} = \oplus \mathcal{H}_{\lambda,\bar{\lambda}} \tag{3.1}$$

Note that the parafermionic current $\phi_b(z)$ has the left charges $\lambda = b$ and the right charge $\bar{\lambda} = 0$. The non-locality of two fields is indicated by the mutual locality exponent γ, which is defined by

$$\gamma\left(\phi_{a,\bar{a}}, \phi_{b,\bar{b}}\right) = \frac{1}{2k}\left(ab - \bar{a}\bar{b}\right). \tag{3.2}$$

So for the currents with zero right charges, the exponent is,

$$\gamma\left(\phi_a, \phi_b\right) = \frac{ab}{2k}. \tag{3.3}$$

From the operator product expansion:

$$\Psi_a(z)\phi_{\lambda,\bar{\lambda}}(w,\bar{w}) = \sum_{-\infty}^{\infty}(z-w)^{-m-1-\frac{a\lambda}{2k}} A_m^{a,\lambda}\phi_{\lambda,\bar{\lambda}}(w,\bar{w}), \tag{3.4}$$

we know that the dimension of $A_m^{a,\lambda}\phi_{\lambda,\bar{\lambda}}(w,\bar{w})$ is

$$\Delta(A_m^{a,\lambda}\phi_{\lambda,\bar{\lambda}}) = h - m - \frac{a\lambda}{2k} - \frac{a^2}{4k}, \tag{3.5}$$

where h is the dimension of the parafermionic field $\phi_{\lambda,\bar{\lambda}}(w,\bar{w})$.

Now we consider the action of the modes of the energy-momentum stress on the highest weight state. The highest weight state is defined by

$$A_n^{a,\Lambda}\phi_{\Lambda,\bar{\Lambda}}^{\Lambda,\bar{\Lambda}}(w,\bar{w}) = 0, \quad \text{for} \quad n > 0, \text{or} \quad n = 0 \quad \text{if} \quad a > 0. \tag{3.6}$$

Then the action of L_m on $\phi_{\lambda,\bar{\lambda}}(w,\bar{w})$ is

$$L_n\phi_{\lambda,\bar{\lambda}}(w,\bar{w}) = \frac{k}{2k+6}\sum_a \left[\left[\Psi_a\Psi_{-a}\right]_{0,n}^{\lambda}\phi_{\lambda,\bar{\lambda}}(w,\bar{w}).\right. \tag{3.7}$$

And so

$$\begin{aligned} L_n\phi_{\Lambda,\bar{\Lambda}}^{\Lambda,\bar{\Lambda}}(w,\bar{w}) &= \left[\frac{\Lambda(\Lambda+4)}{4(k+3)} - \frac{\Lambda^2}{4k} - \frac{7k}{16(k+3)}\right]\phi_{\Lambda,\bar{\Lambda}}^{\Lambda,\bar{\Lambda}}(w,\bar{w}) \\ &= \Delta_\Lambda^\Lambda \phi_{\Lambda,\bar{\Lambda}}^{\Lambda,\bar{\Lambda}}(w,\bar{w}). \end{aligned} \tag{3.8}$$

If we define the state

$$\phi_{\lambda,\bar{\lambda}}^{\Lambda,\bar{\Lambda}}(w,\bar{w}) = \prod_i A_{-n_i}^{a_i,\lambda_i} \prod_j \bar{A}_{-n_j}^{\bar{a}_j,\bar{\lambda}_j} \phi_{\Lambda,\bar{\Lambda}}^{\Lambda,\bar{\Lambda}}(w,\bar{w}), \tag{3.9}$$

then we get

$$\Delta_\lambda^\Lambda = \frac{\Lambda(\Lambda+4)}{4(k+3)} - \frac{\lambda^2}{4k} - \frac{7k}{16(k+3)} + \sum_i n_i, \tag{3.10}$$

where $\lambda = \Lambda + \sum_i a_i$, and $n_i \in \mathbf{Z}$ if $a_i = l$, or $n_i \in \mathbf{Z} + \frac{\mathbf{1}}{\mathbf{2}}$ if $a_i = \tilde{l}$.

4 Free field representation of the twisted parafermions

It was shown in [18] that the twisted affine current algebra $A_2^{(2)}$ allows the following representation in terms of the twisted parafermionic currents:

$$\begin{aligned}
j^+(z) &= 2\sqrt{k}\psi_1(z)e^{\frac{i}{\sqrt{2k}}\phi_0(z)}, \qquad j^-(z) = 2\sqrt{k}\psi_{-1}(z)e^{-\frac{i}{\sqrt{2k}}\phi_0(z)}, \\
j^0(z) &= 2\sqrt{2k}i\partial\phi_0(z), \qquad J^+(z) = 2\sqrt{k}\psi_{\tilde{1}}(z)e^{\frac{i}{\sqrt{2k}}\phi_0(z)}, \\
J^-(z) &= 2\sqrt{k}\psi_{-\tilde{1}}(z)e^{-\frac{i}{\sqrt{2k}}\phi_0(z)}, \qquad J^{++}(z) = 2\sqrt{k}\psi_{\tilde{2}}(z)e^{i\sqrt{\frac{2}{k}}\phi_0(z)}, \\
J^{--}(z) &= 2\sqrt{k}\psi_{-\tilde{2}}(z)e^{-i\sqrt{\frac{2}{k}}\phi_0(z)}, \qquad J^0(z) = 2\sqrt{6k}\psi_{\tilde{0}}(z).
\end{aligned} \tag{4.1}$$

where $\phi_0(z)$ is an $U(1)$ current obeying $\phi_0(z)\phi_0(w) = -ln(z-w)$, and has the modes expansion of $\partial\phi_0(z) = \sum_{n\in Z}\phi_n z^{n+1}$.

We now construct a free field representation of the twisted parafermionic currents. The twisted affine current algebra $A_2^{(2)}$ allows a free field representation in terms of three $(\beta,\ \gamma)$ pairs and one 2-component scalar field [19]. To get a free field representation of the twisted parafermionic currents, one need to projecting out a $U(1)$ current, as is seen from (4.1), and consider the twisted parafermion currents as the operators in the space $A_2^{(2)}/U(1)$. This implies that seven independent scalar fields are needed to realize the twisted parafermion algebra. So, we introduce seven scalar fields, $\phi(z)$ and $\xi_j(z)$, $\eta_j(z)$ $(j = 0,\ 1,\ 2)$, which satisfy the following relations:

$$\begin{aligned}
\xi_i(z)\xi_j(w) = -\delta_{ij}ln(z-w), \quad \eta_i(z)\eta_j(w) = -\delta_{ij}ln(z-w), \\
\phi(z)\phi(w) = -ln(z-w).
\end{aligned}$$

The modes expansions are

$$\partial\chi(z) = \sum_{n\in\mathbf{Z}}\chi_n z^{n+1}, \partial\chi(z) = \sum_{n\in\mathbf{Z}+\frac{\mathbf{1}}{\mathbf{2}}}\chi_n z^{n+1}, \quad \chi = \xi_j, \quad \eta_j\ (j = 1,\ 2) \ \text{ or } \ \phi.$$

The conformal dimension of Ψ_b is $1 - \frac{b^2}{4k}$. So we make the ansatz about the free field representation of the twisted parafermionic currents:

$$\Psi_a(z) = f_a(\xi_i(z), \eta_j(z), \phi(z)) e^{\frac{a}{\sqrt{2k}}\phi(z)}, \tag{4.2}$$

where the factor $e^{\frac{a}{\sqrt{2k}}\phi(z)}$ will contribute $-\frac{a^2}{4k}$ to the conformal dimension of $\Psi_a(z)$, and $f_a(\xi_i(z), \eta_j(z), \phi(z))$ are operators with conformal dimension one. Note that the index a and $\tilde{a}$ are different, but as a numbers, they are the same. From dimensional analysis, no term of the form $e^{\frac{a}{\sqrt{2k}}\phi(z)}$ will appear in $f_b(\xi_i(z), \eta_j(z), \phi(z))$. Notice that $(\xi_i(z) - i\eta_i(z))(\xi_j(w) - i\eta_j(w))$ have no contribution to the OPE. We find, after a long and tedious calculation,

$$\begin{aligned}
f_1(z) &= \frac{1}{2\sqrt{2k}}\left[-\alpha_0\partial\xi_0(z) - 2\partial\xi_1(z) + \partial\xi_2(z) + \partial\phi(z)\right] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[-\alpha_0(\xi_0(z) - i\eta_0(z)) \\
&\qquad -2(\xi_1(z) - i\eta_1(z)) + (\xi_2(z) - i\eta_2(z))]\}, \\
f_{-1}(z) &= \frac{1}{2\sqrt{2k}}\{-[(4k+1)[\alpha_0(\partial\xi_0(z) - i\partial\eta_0(z)) + 2(\partial\xi_1(z) - i\partial\eta_1(z)) \\
&\qquad -(\partial\xi_2(z) - i\partial\eta_2(z)) - \partial\phi(z)] \\
&\quad + i\left[\alpha_0\partial\eta_0(z) + 2\partial\eta_1(z) - \partial\eta_2(z)\right]] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[\alpha_0(\xi_0(z) - i\eta_0(z)) + 2(\xi_1(z) - i\eta_1(z)) \\
&\qquad -(\xi_2(z) - i\eta_2(z))]\} \\
&\quad + 3\left[\alpha_0\partial\xi_0(z) + 2\partial\xi_1(z) - \partial\xi_2(z) - \partial\phi(z)\right] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[-(4+\alpha_0)(\xi_0(z) - i\eta_0(z)) \\
&\qquad -2(1-\alpha_0)(\xi_1(z) - i\eta_1(z)) + 3(\xi_2(z) - i\eta_2(z))]\} \\
&\quad + 2\left[2\partial\xi_0(z) - \alpha_0\partial\xi_1(z) - \partial\xi_2(z) + \partial\phi(z)\right] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[3(\xi_0(z) - i\eta_0(z)) - (\alpha_0+1)(\xi_1(z) - i\eta_1(z)) \\
&\qquad -(1-\alpha_0)(\xi_2(z) - i\eta_2(z))]\} \\
&\quad - 4\left[\partial\xi_0(z) - \partial\xi_1(z) + \alpha_0\partial\xi_2(z) - 2\partial\phi(z)\right] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[-7(\xi_0(z) - i\eta_0(z)) + (1+3\alpha_0)(\xi_1(z) - i\eta_1(z)) \\
&\qquad +(3-\alpha_0)(\xi_2(z) - i\eta_2(z))]\} \\
&\quad + 2\sqrt{3}i\alpha_0\left[\partial\xi_0(z) + \partial\xi_1(z) + 2\partial\xi_2(z) + \alpha_0\partial\phi(z)\right] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[-2(\xi_0(z) - i\eta_0(z)) + \alpha_0(\xi_1(z) - i\eta_1(z)) \\
&\qquad +(\xi_2(z) - i\eta_2(z))]\}\}, \\
f_{\tilde{2}}(z) &= \frac{1}{2\sqrt{2k}}\left[-\partial\xi_0(z) + \partial\xi_1(z) - \alpha_0\partial\xi_2(z) + 2\partial\phi(z)\right] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[-(\xi_0(z) - i\eta_0(z)) + (\xi_1(z) - i\eta_1(z)) \, . \\
&\qquad -\alpha_0(\xi_2(z) - i\eta_2(z))]\},
\end{aligned}$$

$$
\begin{aligned}
f_{\tilde{1}}(z) &= \frac{1}{2\sqrt{2}k}\{[2\partial\xi_0(z) - \alpha_0\partial\xi_1(z) - \partial\xi_2(z) + \partial\phi(z)] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[2(\xi_0(z) - i\eta_0(z)) - \alpha_0(\xi_1(z) - i\eta_1(z)) \\
&\qquad -(\xi_2(z) - i\eta_2(z))]\} \\
&\quad + 2[\partial\xi_0(z) - \partial\xi_1(z) + \alpha_0\partial\xi_2(z) - 2\partial\phi(z)] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[-(1-\alpha_0)(\xi_0(z) - i\eta_0(z)) + 3(\xi_1(z) - i\eta_1(z)) \\
&\qquad -(1+\alpha_0)(\xi_2(z) - i\eta_2(z))]\}\}, \\
f_{\tilde{0}}(z) &= \frac{\sqrt{3}}{2k}\{[-\alpha_0\partial\xi_0(z) - 2\partial\xi_1(z) + \partial\xi_2(z) + \partial\phi(z)] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[-(2+\alpha_0)(\xi_0(z) - i\eta_0(z)) \\
&\qquad -(2-\alpha_0)(\xi_1(z) - i\eta_1(z)) + 2(\xi_2(z) - i\eta_2(z))]\} \\
&\quad + [2\partial\xi_0(z) - \alpha_0\partial\xi_1(z) - \partial\xi_2(z) + \partial\phi(z)] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[(2+\alpha_0)(\xi_0(z) - i\eta_0(z)) \\
&\qquad +(2-\alpha_0)(\xi_1(z) - i\eta_1(z)) - 2(\xi_2(z) - i\eta_2(z))]\} \\
&\quad + [\partial\xi_0(z) - \partial\xi_1(z) + \alpha_0\partial\xi_2(z) - 2\partial\phi(z)] \\
&\quad \times [\exp\{\frac{1}{\sqrt{2k}}[-(1-2\alpha_0)(\xi_0(z) - i\eta_0(z)) \\
&\qquad +5(\xi_1(z) - i\eta_1(z)) - (2+\alpha_0)(\xi_2(z) - i\eta_2(z))]\} \\
&\quad\; + \exp\{\frac{1}{\sqrt{2k}}[-5(\xi_0(z) - i\eta_0(z)) + (1+2\alpha_0)(\xi_1(z) - i\eta_1(z)) \\
&\qquad +(2-\alpha_0)(\xi_2(z) - i\eta_2(z))]\}] \\
&\quad - \frac{i\alpha_0}{\sqrt{3}}(\partial\xi_0(z) + \partial\xi_1(z) + 2\partial\xi_2(z) + \alpha_0\partial\phi(z))\}. \\
f_{-\tilde{1}}(z) &= \frac{1}{2\sqrt{2}k}\{-[(4k+5)[2\alpha_0(\partial\xi_0(z) - i\partial\eta_0(z)) - \alpha_0(\partial\xi_1(z) - i\partial\eta_1(z)) \\
&\qquad -(\partial\xi_2(z) - i\partial\eta_0(z)) + \partial\phi(z)] \\
&\quad -(2-4\alpha_0)i\partial\eta_0(z) + (12-\alpha_0)i\partial\eta_1(z) - (5+4\alpha_0)i\partial\eta_2(z)] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[-2(\xi_0(z) - i\eta_0(z)) + \alpha_0(\xi_1(z) - i\eta_1(z)) \\
&\qquad +(\xi_2(z) - i\eta_2(z))]\} \\
&\quad + 2[\alpha_0\partial\xi_0(z) + 2\partial\xi_1(z) - \partial\xi_2(z) - \partial\phi(z)] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[(1-\alpha_0)(\xi_0(z) - i\eta_0(z)) - 3(\xi_1(z) - i\eta_1(z)) \\
&\qquad +(1+\alpha_0)(\xi_2(z) - i\eta_2(z))]\} \\
&\quad + 3[-2\partial\xi_0(z) + \alpha_0\partial\xi_1(z) + \partial\xi_2(z) - \partial\phi(z)] \\
&\quad \times \exp\{\frac{1}{\sqrt{2k}}[2(1+\alpha_0)(\xi_0(z) - i\eta_0(z)) \\
&\qquad +(4-\alpha_0)(\xi_1(z) - i\eta_1(z)) - 3(\xi_2(z) - i\eta_2(z))]\} \\
&\quad - 2[\partial\xi_0(z) - \partial\xi_1(z) + \alpha_0\partial\xi_2(z) - 2\partial\phi(z)]
\end{aligned}
$$

$$
\begin{aligned}
& \times [\exp \{\frac{1}{\sqrt{2k}} [-(1-3\alpha_0)(\xi_0(z)-i\eta_0(z)) \\
& \qquad +7(\xi_1(z)-i\eta_1(z))-(3+\alpha_0)(\xi_2(z)-i\eta_2(z))]\} \\
& \quad + 3\ \exp\{\frac{1}{\sqrt{2k}} [-(5-\alpha_0)(\xi_0(z)-i\eta_0(z)) \\
& \qquad + (3+2\alpha_0)(\xi_1(z)-i\eta_1(z)) \\
& \qquad +(1-\alpha_0)(\xi_2(z)-i\eta_2(z))]\}] \\
& + 2\sqrt{3}i\alpha_0 [\partial\xi_0(z)+\partial\xi_1(z)+2\partial\xi_2(z)+\alpha_0\partial\phi(z)] \\
& \times \exp\{\frac{1}{\sqrt{2k}} [\alpha_0(\xi_0(z)-i\eta_0(z))+2(\xi_1(z)-i\eta_1(z)) \\
& \qquad -(\xi_2(z)-i\eta_2(z))]\}\},
\end{aligned}
$$

$$
\begin{aligned}
f_{-\tilde{2}}(z) \;=\; \frac{1}{2\sqrt{2k}} & \{-4 [(k+1) [(\partial\xi_0(z)-i\partial\eta_0(z))-(\partial\xi_1(z)-i\partial\eta_1(z)) \\
& \qquad +\alpha_0(\partial\xi_2(z)-i\partial\eta_2(z))-2\partial\phi(z)] \\
& + i [\partial\eta_0(z)-\partial\eta_1(z)+\alpha_0\partial\eta_2(z)]] \\
& \times \exp\{\frac{1}{\sqrt{2k}} [(\xi_0(z)-i\eta_0(z))-(\xi_1(z)-i\eta_1(z)) \\
& \qquad +\alpha_0(\xi_2(z)-i\eta_2(z))]\} \\
& + 2 [-\alpha_0\partial\xi_0(z)-2\partial\xi_1(z)+\partial\xi_2(z)+\partial\phi(z)] \\
& \times \exp\{\frac{1}{\sqrt{2k}} [-(6+\alpha_0)(\xi_0(z)-i\eta_0(z)) \\
& \qquad -(2-3\alpha_0)(\xi_1(z)-i\eta_1(z))+4(\xi_2(z)-i\eta_2(z))]\} \\
& - 2 [(2-\alpha_0)i\partial\eta_0(z)-(6-\alpha_0)i\partial\eta_1(z)+(2+4\alpha_0)i\partial\eta_2(z) \\
& \quad + (4k+2) [-2(\partial\xi_0(z)-i\partial\eta_0(z))+\alpha_0(\partial\xi_1(z)-i\partial\eta_1(z)) \\
& \qquad + (\partial\xi_2(z)-i\partial\eta_2(z))]-4(k+2)\partial\phi(z) \\
& \quad - 3(2-\alpha_0)(\partial\xi_0(z)-i\partial\eta_0(z)) \\
& \quad +3(2+\alpha_0)\alpha_0(\partial\xi_1(z)-i\partial\eta_1(z))] \\
& \times \exp\{\frac{1}{\sqrt{2k}} [-(2-\alpha_0)(\xi_0(z)-i\eta_0(z)) \\
& \qquad +(2+\alpha_0)(\xi_1(z)-i\eta_1(z))]\} \\
& + 2 [-2\partial\xi_0(z)+\alpha_0\partial\xi_1(z)+\partial\xi_2(z)-\partial\phi(z)] \\
& \times \exp\{\frac{1}{\sqrt{2k}} [(2+3\alpha_0)(\xi_0(z)-i\eta_0(z)) \\
& \qquad +(6-\alpha_0)(\xi_1(z)-i\eta_1(z))-4(\xi_2(z)-i\eta_2(z))]\} \\
& - [\partial\xi_0(z)-\partial\xi_1(z)+\alpha_0\partial\xi_2(z)-2\partial\phi(z)] \\
& \times [\exp\{\frac{1}{\sqrt{2k}} [-(1-4\alpha_0)(\xi_0(z)-i\eta_0(z)) \\
& \qquad +9(\xi_1(z)-i\eta_1(z))-(4+\alpha_0)(\xi_2(z)-i\eta_2(z))]\} \\
& \quad - 3\ \exp\{\frac{1}{\sqrt{2k}} [-9(\xi_0(z)-i\eta_0(z))+(1+4\alpha_0)(\xi_1(z)-i\eta_1(z)) \\
& \qquad +(4-\alpha_0)(\xi_2(z)-i\eta_2(z))]\}
\end{aligned}
$$

$$
\begin{aligned}
&+6\exp\{\frac{1}{\sqrt{2k}}[-(5-2\alpha_0)(\xi_0(z)-i\eta_0(z))\\
&\qquad +(5+2\alpha_0)(\xi_1(z)-i\eta_1(z))-\alpha_0(\xi_2(z)-i\eta_2(z))]\}]\\
&+2\sqrt{3}i\alpha_0\,[\partial\xi_0(z)+\partial\xi_1(z)+2\partial\xi_2(z)+\alpha_0\partial\phi(z)]\\
&\times[\exp\{\frac{2}{\sqrt{2k}}[\alpha_0(\xi_0(z)-i\eta_0(z))+2(\xi_1(z)-i\eta_1(z))\\
&\qquad -(\xi_2(z)-i\eta_2(z)]\}\\
&-\exp\{\frac{2}{\sqrt{2k}}[-2(\xi_0(z)-i\eta_0(z))+\alpha_0(\xi_1(z)-i\eta_1(z))\\
&\qquad -(\xi_2(z)-i\eta_2(z))]\}]\}\,,
\end{aligned}
\tag{4.3}
$$

where $\alpha_0=\sqrt{-2(k+3)}$. It can be checked that the $\Psi_a(z)$ satisfy all of the OPE relations of equation (2.2). By the above representation of the twisted parafermionic currents, we can bosonize the parafermion energy-momentum tensor T_ψ. The result is

$$
\begin{aligned}
T_\psi(z) &= -\frac{1}{2}:\{(\partial\xi_0(z))^2+(\partial\xi_1(z))^2+(\partial\xi_2(z))^2\\
&\quad +(\partial\eta_0(z))^2+(\partial\eta_1(z))^2+(\partial\eta_2(z))^2+(\partial\phi(z))^2\}:\\
&-\frac{1}{2\sqrt{2k}}\{-(1-\alpha_0)(\partial^2\xi_0(z)-i\partial^2\eta_0(z))+(1+\alpha_0)(\partial^2\xi_1(z)-i\partial^2\eta_1(z))\\
&\quad +\alpha_0(\partial^2\xi_2(z)-i\partial^2\eta_2(z))-4\partial^2\phi(z)\}\\
&+\frac{1}{\sqrt{k(k+3)}}\left(\partial^2\eta_0(z)+\partial^2\eta_1(z)+\partial^2\eta_2(z)\right).
\end{aligned}
\tag{4.4}
$$

It is a easy work to check that the central charge of this operator is indeed $c_{\rm tpf}$.

5 Spin-2 Primary field and novel algebraic structure

In conformal field theory, primary fields are fundamental objects. The descendant fields can be obtained from primary fields.

It is well know that, the energy-momentum stress is not primary field (unless the central charge is zero). The lowest spin of the primary field obtained by the Hamiltonian reduction approach is three [20, 21]. This agrees with the other methods, such as high order Casimir, coset model [22], free field realization [23], or the parafermion construction [15]. For more references about W-algebras see the reviews [21, 24]. In the following, we use the twisted parafermionic currents to construct a primary field of spin two. Define the spin-2 currents:

$$
\begin{aligned}
\tilde{w}_2(z) &= \frac{k(4k-1)}{4(2k+3)(k-1)}\sum_a[\Psi_a\Psi_{-\tilde{a}}]_0\\
&= \frac{k(4k-1)}{4(2k+3)(k-1)}\sum_{l=\pm1,\pm\tilde{1}}[\Psi_l\Psi_{-\tilde{l}}]_0.
\end{aligned}
\tag{5.1}
$$

It should be understood that $\Psi_{\pm 2}(z) = \Psi_0(z) \equiv 0$, and $\tilde{\tilde{a}} = a$. The action of $\tilde{w}_2(z)$ on $\Psi_b(z)$ is given by the following OPE:

$$\tilde{w}_2(z)\Psi_b(w) = \frac{1 - b^2/4k}{(z-w)^2}\Psi_{\tilde{b}}(w) + \frac{1}{z-w}\partial\Psi_{\tilde{b}}(w) + \ldots. \tag{5.2}$$

This OPE fixes the normalization of $\tilde{w}_2(z)$. (5.2) suggests that $\tilde{w}_2(z)$ behaves similar to a energy-momentum tensor except that it transforms Ψ_b into $\Psi_{\tilde{b}}$ and vice versa. Recall that Ψ_b and $\Psi_{\tilde{b}}$ have the same conformal dimensions. The OPEs of $\tilde{w}_2(z)$ with itself and the stress tensor are

$$\begin{aligned} \tilde{w}_2(z)\tilde{w}_2(w) &= \frac{\tilde{c}/2}{(z-w)^4} + \frac{2U(w)}{(z-w)^2} + \frac{\partial U(w)}{(z-w)} + \cdots. \\ T_\psi(z)\tilde{w}_2(w) &= \frac{2\tilde{w}_2(w)}{(z-w)^2} + \frac{\partial\tilde{w}_2(w)}{z-w} + \ldots, \end{aligned} \tag{5.3}$$

where,

$$\tilde{c} = \frac{(4k-1)^2}{2(k-1)(2k+3)} = \frac{(13c_{\text{tpf}}+5)^2}{24(c_{\text{tpf}}+9)c_{\text{tpf}}-1)}, \tag{5.4}$$

is the central charge of $\tilde{w}_2(z)$, and c_{tpf} is the central charge of the twisted parafermionic energy-momentum tensor, and $U(z)$ is a spin two field given by

$$\begin{aligned} U(z) &= \frac{(k+1)(k+3)(4k-1)^2}{4(2k+3)^2(k-1)^2}\left(T_\psi(z) - \frac{1}{2}\Omega_1(z) - \frac{k(k+2)}{(k+1)(k+3)}\Omega_2(z)\right), \\ \Omega_1(z) &= \frac{1}{2}\left[\psi_{\tilde{0}}\psi_{\tilde{0}}\right]_0(z), \quad \Omega_2(z) = \frac{k}{k+2}\left[\psi_{\tilde{2}}\psi_{-\tilde{2}}\right]_0(z). \end{aligned} \tag{5.5}$$

From (5.3), we see that the field $\tilde{w}_2$ introduced above is a primary field with conformal spin two. Note that the modes expansion of $\tilde{w}_2(z)$ is

$$\tilde{w}_2(z) = \sum_{n \in \mathbf{Z}+\frac{1}{2}} \tilde{w}_n z^{-n-2}. \tag{5.6}$$

So like the spin-1 primary field $\psi_{\tilde{0}}(z)$, $\tilde{w}_2(z)$ lives in the Neveu-Schwarz sector, and it transforms fields in the Ramond sector into those in the Neveu-Schwarz sector or vice versa. If we regard the spin-2 currents as affine connections, then the energy-momentum tensor is a project connection that transforms fields in one sector, while field $\tilde{w}_2(z)$ is a project connection which transforms a field in one sector into a field in a different sector.

From the above, we see that $\tilde{w}_2(z)$, $T(z)$ do not close to an algebra. It can be checked that the field $U(z)$ with itself also cannot form a closed algebra. But if we decompose the field $U(z)$ as above, then the algebra generated by $\tilde{w}_2(z)$, $T(z)$, $\Omega_1(z)$, $\Omega_2(z)$ is closed, as can be seen from the following additional OPEs,

$$
\begin{aligned}
\Omega_1(z)\Omega_1(w) &= \frac{1/2}{(z-w)^4} + \frac{2\Omega_1(w)}{(z-w)^2} + \frac{\partial\Omega_1(w)}{(z-w)} + \cdots, \\
\Omega_2(z)\Omega_2(w) &= \frac{(k-1)/(k+2)}{(z-w)^4} + \frac{2\Omega_2(w)}{(z-w)^2} + \frac{\partial\Omega_2(w)}{(z-w)} + \cdots, \\
\Omega_1(z)\Omega_2(w) &= \cdots, \\
T_\psi(z)\Omega_1(w) &= \frac{1/2}{(z-w)^4} + \frac{2\Omega_1(w)}{(z-w)^2} + \frac{\partial\Omega_1(w)}{(z-w)} + \cdots, \\
T_\psi(z)\Omega_2(w) &= \frac{(k-1)/(k+2)}{(z-w)^4} + \frac{2\Omega_2(w)}{(z-w)^2} + \frac{\partial\Omega_2(w)}{(z-w)} + \cdots, \\
\Omega_1(z)\tilde{w}_2(w) &= \cdots, \\
\Omega_2(z)\tilde{w}_2(w) &= \frac{1}{k+2}\left\{\frac{2\tilde{w}_2(w)}{(z-w)^2} + \frac{\partial\tilde{w}_2(w)}{(z-w)} + \cdots\right\}.
\end{aligned}
$$

So the situation here is similar to that in W-algebra, where for instance $W_3(z)$, $T(z)$ are not closed since a spin-4 field will appear in the OPE of $W_3(z)$ with itself. The spin-4 field is also decomposed into two fields which together with $T(z)$ and $W_3(z)$ do close to the so-called W_3 algebra [20].

Some remarks are in order. To our knowledge, the algebraic structure we found is new. It is similar to a W-algebra structure, but now all the fields have conformal dimension 2. The primary field $\tilde{w}_2(z)$ is a "spin-2 analog" of W-currents, and it transforms between fields Ψ_a (fields in Ramond sector) and $\Psi_{\tilde{a}}$ (fields in Neveu-Schwaz sector). In a sense, the primary field $\tilde{w}_2(z)$, together with the other two spin-2 fields $\Omega_1(z)$ and $\Omega_2(z)$, "non-abelianizes" the Virasoro algebra generated by $T(z)$. Possible applications of this new algebraic structure are under investigation.

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